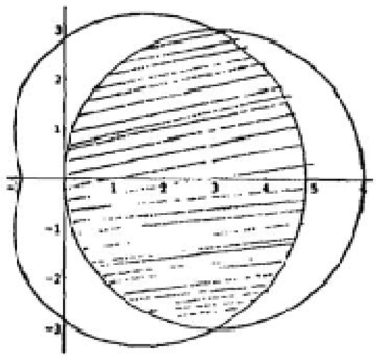
HOSSAM GHANEM

(42) 11.4 Areas And Lengths In Polar Coordinates

Example 1

23 January 2005 A

Let
$r_1 = 6\cos\theta$ and
$r_2 = 2\sqrt{2} + 2\cos\theta$,
where $0 \le \theta \le 2\pi$.
Compute the shaded area



Solution

Intersection point

$6\cos\theta = 2\sqrt{2} + 2\cos\theta$

$4\cos\theta = 2\sqrt{2}$

$\cos\theta = \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4}$

$$A = \frac{1}{2}\int_a^b r^2 d\theta$$

$$A = 2\cdot\frac{1}{2}\int_0^{\frac{\pi}{4}} \left(2\sqrt{2} + 2\cos\theta\right)^2 d\theta + 2\cdot\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (6\cos\theta)^2\, d\theta$$

$$A = \int_0^{\frac{\pi}{4}} \left(8 + 8\sqrt{2}\cos\theta + 4\cos^2\theta\right) d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (36\cos^2\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(8 + 8\sqrt{2}\cos\theta + 2 + 2\cos 2\theta\right) d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (18 + 18\cos 2\theta)\, d\theta$$

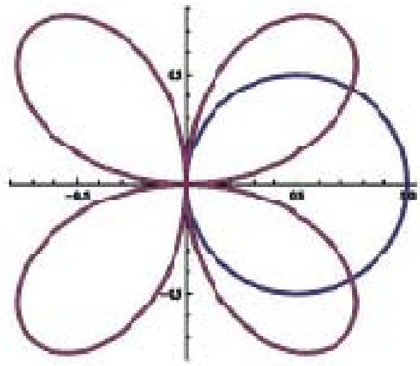
$$= \left[10\,\theta + 8\sqrt{2}\sin\theta + \sin 2\theta\right]_0^{\frac{\pi}{4}} + \left[18\,\theta + 9\sin 2\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{10}{4}\pi + 8 + 1 - 0 + \frac{18}{2}\pi + 0 + -\left(\frac{18}{4}\pi + 9\right) = \left(\frac{10 + 36 - 18}{4}\right)\pi + (8 + 1 - 9) = 7\pi$$

Example 2

32 January 2009 A

The figure below shows the graphs of the polar equations $r = \cos\theta$ and $r = \sin 2\theta$. Find the area of the region that lies inside both curves.



Solution

Intersection point

$$\sin 2\theta = \cos\theta$$
$$2\sin\theta\cos\theta - \cos\theta = 0$$
$$\cos\theta\,(2\sin\theta - 1)$$
$$\sin\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{6}$$

$$A = \frac{1}{2}\int_a^b r^2\,d\theta$$

$$A = 2\cdot\frac{1}{2}\int_0^{\frac{\pi}{6}} (\sin 2\theta)^2\,d\theta + 2\cdot\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$A = \int_0^{\frac{\pi}{6}} \left(\frac{1}{2} - \frac{1}{2}\cos 4\theta\right) d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \left[\frac{1}{2}\theta - \frac{1}{8}\sin 4\theta\right]_0^{\frac{\pi}{6}} + \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{12} - \frac{1}{8}\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{4} + 0 - \left(\frac{\pi}{12} + \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)\right)$$

$$= \left(\frac{1}{12} + \frac{1}{4} - \frac{1}{12}\right)\pi + \left(\frac{-1}{8} + \frac{-1}{4}\right)\frac{\sqrt{3}}{2} = \frac{\pi}{4} - \frac{3}{16}\sqrt{3}$$

Example 3

39 June 4 , 2011

Let C be the polar curve $r = \sqrt{\theta}\cos\theta, \qquad 0 \le \theta \le \frac{\pi}{2}$

(a) Sketch the curve. (1 pts)

(b) Find the equation of the tangent line to C at the point corresponding to $\theta = \frac{\pi}{4}$ (2 pts)

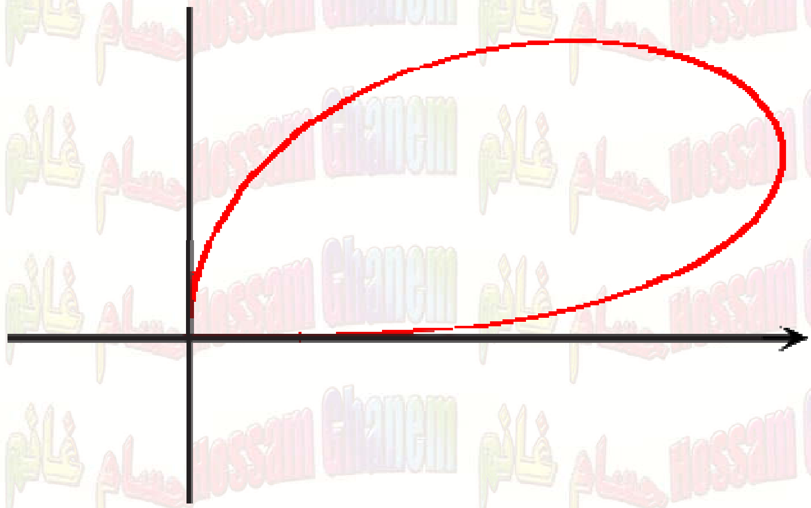
(c) Find the area enclosed by C (2 pts)

Solution

(a)

		θ	r
0		0	0
1		$\frac{\pi}{2}$	0

	θ	r
1	$0 \to \frac{\pi}{2}$	$0 \to 0$



(b)

p

$r = \sqrt{\theta}\cos\theta$

$r|_{\theta=\frac{\pi}{4}} = \sqrt{\frac{\pi}{4}}\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{\pi}}{2}\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{\pi}}{2\sqrt{2}}$

$x = r\cos\theta = \frac{\sqrt{\pi}}{2\sqrt{2}}\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{\pi}}{4}$

$y = r\sin\theta = \frac{\sqrt{\pi}}{2\sqrt{2}}\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{\pi}}{4}$

$$p\left(\frac{\sqrt{\pi}}{4}, \frac{\sqrt{\pi}}{4}\right)$$

m

$$\frac{dr}{d\theta} = \frac{1}{2\sqrt{\theta}}\cos\theta - \sqrt{\theta}\sin\theta$$

$$\left.\frac{dr}{d\theta}\right|_{\theta=\frac{\pi}{4}} = \frac{1}{2\sqrt{\frac{\pi}{4}}}\cos\frac{\pi}{4} - \sqrt{\frac{\pi}{4}}\sin\frac{\pi}{4} = \frac{1}{2\cdot\frac{\sqrt{\pi}}{2}}\cdot\frac{1}{\sqrt{2}} - \frac{\sqrt{\pi}}{2}\cdot\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}\sqrt{\pi}} - \frac{\sqrt{\pi}}{2\sqrt{2}} = \frac{2-\pi}{2\sqrt{2}\sqrt{\pi}}$$

$$m = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\frac{2-\pi}{2\sqrt{2}\sqrt{\pi}}\cdot\frac{1}{\sqrt{2}} + \frac{\sqrt{\pi}}{2\sqrt{2}}\cdot\frac{1}{\sqrt{2}}}{\frac{2-\pi}{2\sqrt{2}\sqrt{\pi}}\cdot\frac{1}{\sqrt{2}} - \frac{\sqrt{\pi}}{2\sqrt{2}}\cdot\frac{1}{\sqrt{2}}} = \frac{\frac{2-\pi}{4\sqrt{\pi}} + \frac{\sqrt{\pi}}{4}}{\frac{2-\pi}{4} - \frac{\sqrt{\pi}}{4}} = \frac{2-\pi+\pi}{2-\pi-\pi} = \frac{2}{2-2\pi} = \frac{1}{1-\pi}$$

بالضرب في $4\sqrt{\pi}$ بسط و مقام

$$m = \frac{1}{1-\pi} \qquad p\left(\frac{\sqrt{\pi}}{4}, \frac{\sqrt{\pi}}{4}\right)$$

$$y - y_1 = m(x - x_1)$$

$$y - \frac{\sqrt{\pi}}{4} = \frac{1}{1-\pi}\left(x - \frac{\sqrt{\pi}}{4}\right)$$

(c)

$$A = \frac{1}{2}\int_a^b r^2\,d\theta$$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\sqrt{\theta}\,\cos\theta\right)^2 d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \theta\cos^2\theta\,d\theta = \frac{1}{4}\int_0^{\frac{\pi}{2}} \theta(1+\cos 2\theta)d\theta$$

$$u = \frac{1}{4}\theta \qquad dv = 1 + \cos 2\theta$$

$$du = \frac{1}{4}d\theta \qquad v = \theta + \frac{1}{2}\sin 2\theta$$

$$I = uv - \int v\,du$$

$$A = \frac{1}{4}\left[\theta\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{\frac{\pi}{2}} - \frac{1}{4}\int_0^{\frac{\pi}{2}}\left(\theta + \frac{1}{2}\sin 2\theta\right)d\theta$$

$$= \frac{1}{4}\left[\frac{\pi}{2}\left(\frac{\pi}{2} + 0\right) - 0\right] - \frac{1}{4}\left[\frac{1}{2}\theta^2 - \frac{1}{4}\cos 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left(\frac{\pi^2}{4}\right) - \frac{1}{4}\left[\frac{1}{2}\cdot\frac{\pi^2}{4} - \frac{1}{4}(-1) - \left(0 - \frac{1}{4}\right)\right] = \frac{\pi^2}{16} - \frac{\pi^2}{32} - \frac{1}{16} - \frac{1}{16} = \frac{1}{32}(2\pi^2 - \pi^2 - 2 - 2)$$

$$= \frac{1}{32}(\pi^2 - 4)$$

$1-\sin\theta=\left(\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)\right)^2$	$1-\cos\theta=2\sin^2\left(\frac{\theta}{2}\right)$
$1+\sin\theta=\left(\cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)\right)^2$	$1+\cos\theta=2\cos^2\left(\frac{\theta}{2}\right)$

The length of a curve

$$L=\int_a^b\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

Example 4 Find the length of the polar curve $r=\sin\theta+\cos\theta$, where $0\le\theta\le2\pi$ 23 January 2005 A

Solution

$$r=\sin\theta+\cos\theta$$

$$\frac{dr}{d\theta}=\cos\theta-\sin\theta$$

$$r^2=\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta=1+\sin2\theta$$

$$\left(\frac{dr}{d\theta}\right)^2=\cos^2\theta-2\sin\theta\cos\theta\ \sin^2\theta=1-\sin2\theta$$

$$r^2+\left(\frac{dr}{d\theta}\right)^2=1+\sin2\theta+1-\sin2\theta=2$$

$$L=\int_a^b\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$=\int_0^{2\pi}\sqrt{2}\,d\theta=\sqrt{2}\Big[\theta\Big]_0^{2\pi}=\sqrt{2}(2\pi-0)=2\sqrt{2}\,\pi$$

Example 5 Find the length of the curve $r = 4\cos^2\left(\frac{\theta}{2}\right)$ 27 June 2006 A

Solution

$r = 4\cos^2\left(\frac{\theta}{2}\right) = 4\cdot\frac{1}{2}(1+\cos\theta) = 2+2\cos\theta$

$\frac{dr}{d\theta} = -2\sin\theta$

$r^2 = 4+8\cos\theta+4\cos^2\theta$

$\left(\frac{dr}{d\theta}\right)^2 = 4\sin^2\theta$

$r^2+\left(\frac{dr}{d\theta}\right)^2 = 4+8\cos\theta+4\cos^2\theta+4\sin^2\theta$

$= 4+8\cos\theta+4(\cos^2\theta+\sin^2\theta) = 4+8\cos\theta+4 = 8+8\cos\theta$

$= 8(1+\cos\theta) = 8\cdot 2\cos^2\frac{\theta}{2} = 16\cos^2\frac{\theta}{2}$

$\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} = 4\cos\frac{\theta}{2}$

$L = \int_a^b \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi}\left(4\cos\frac{\theta}{2}\right)d\theta = 2\int_0^{\pi}\left(4\cos\frac{\theta}{2}\right)d\theta = 2\cdot 4\cdot 2\left[\sin\frac{\theta}{2}\right]_0^{\pi} = 16(1-0) = 16$

Example 6 25 August 2005 A

Sketch the cardioid $r = 2-2\sin\theta$ and the circle $r = 2\sin\theta$ and label their points of Intersection with the axes.

a. Show that the cardioid has horizontal tangent lines at $(1,\pi/6)$ and $(1,5\pi/6)$.

b. Find the arc length of the part of the cardioid corresponding to $0\le\theta\le\pi/2$.

c. Find the area of the region that is inside the circle and outside the cardioids.

Solution

(a)

$r = 2-2\sin\theta$

$\frac{dr}{d\theta} = -2\cos\theta$

$m = \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$

$\therefore$ H.T at $\frac{dr}{d\theta}\sin\theta + r\cos\theta = 0$

$(-2\cos\theta)\sin\theta + (2-2\sin\theta)\cos\theta = 0$

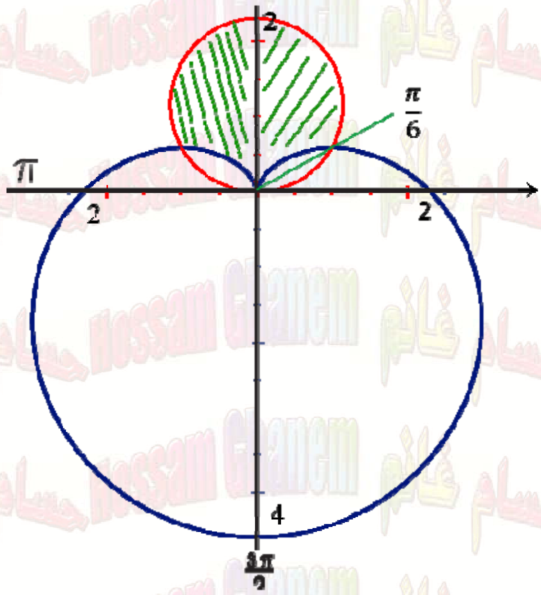
$\cos\theta\,[-2\sin\theta+2-2\sin\theta] = 0$

$\cos\theta\,[2-4\sin\theta] = 0$

$2\cos\theta\,[1-2\sin\theta] = 0$

$\sin\theta = \frac{1}{2}$

$\alpha = \frac{\pi}{6}$



$\boxed{1}$ $\theta = \alpha = \frac{\pi}{6}$ $\boxed{2}$ $\theta = \pi - \alpha = \pi - \frac{\pi}{6} = -\frac{5\pi}{6}$

at $\theta = \frac{\pi}{6}$

$r = 2 - 2\sin\frac{\pi}{6} = 2 - 2\left(\frac{1}{2}\right) = 2 - 1 = 1$

at $\theta = \frac{5\pi}{6}$

$r = 2 - 2\sin\frac{5\pi}{6} = 2 - 2\left(\frac{1}{2}\right) = 2 - 1 = 1$

$\therefore$ H.T at $\left(1, \frac{\pi}{6}\right)$ and $\left(1, \frac{5\pi}{6}\right)$

(b)

$r = 2 - 2\sin\theta$

$\frac{dr}{d\theta} = -2\cos\theta$

$r^2 = 4 - 8\sin\theta + 4\sin^2\theta$

$\left(\frac{dr}{d\theta}\right)^2 = 4\cos^2\theta$

$\left(\frac{dr}{d\theta}\right)^2 + r^2 = 4 - 8\sin\theta + 4\sin^2\theta + 4\cos^2\theta = 4 - 8\sin\theta + 4(\sin^2\theta + \cos^2\theta)$

$= 8 - 8\sin\theta = 8(1 - \sin\theta) = 8\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2$

$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \sqrt{8}\int_0^{\frac{\pi}{2}} \left|\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right| d\theta = 2\sqrt{2}\left[2\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right]_0^{\frac{\pi}{2}}$

$= 2\sqrt{2}\left[2\cdot\frac{1}{\sqrt{2}} + 2\cdot\frac{1}{\sqrt{2}} - (0+2)\right] = 4 + 4 - 4\sqrt{2} = 8 - 4\sqrt{2}$

Intersection point

$2\sin\theta = 2 - 2\sin\theta$

$4\sin\theta = 2$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

$A = \frac{1}{2}\int_a^b r^2\, d\theta$

$A = 2\cdot\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2\sin\theta)^2 - 2\cdot\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2 - 2\sin\theta)^2\, d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\sin^2\theta - (4 - 8\sin\theta + 4\sin^2\theta)\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (8\sin\theta - 4)\, d\theta = \Big[-8\cos\theta - 4\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= 0 - 4\left(\frac{\pi}{2}\right) - 8\cdot\frac{\sqrt{3}}{2} - 4\left(\frac{\pi}{6}\right) = -2\pi + 8\cdot\frac{\sqrt{3}}{2} + \frac{2}{3}\pi = 4\sqrt{3} - \frac{4\pi}{3}$

Example 7	(2 + 2 pts.) Consider the curve given by the polar equation $r = (\pi - \theta)^2 \quad ; \quad 0 \le \theta \le 2\pi$ 38 Jan. 22, 2011 (a) Sketch the curve. (b) Find the area inside the curve

Solution

		θ	r
0		0	π^2
1		$\frac{\pi}{2}$	$\frac{\pi^2}{4}$
2		π	0
3		$\frac{3\pi}{2}$	$\frac{\pi^2}{4}$
4		2π	π^2

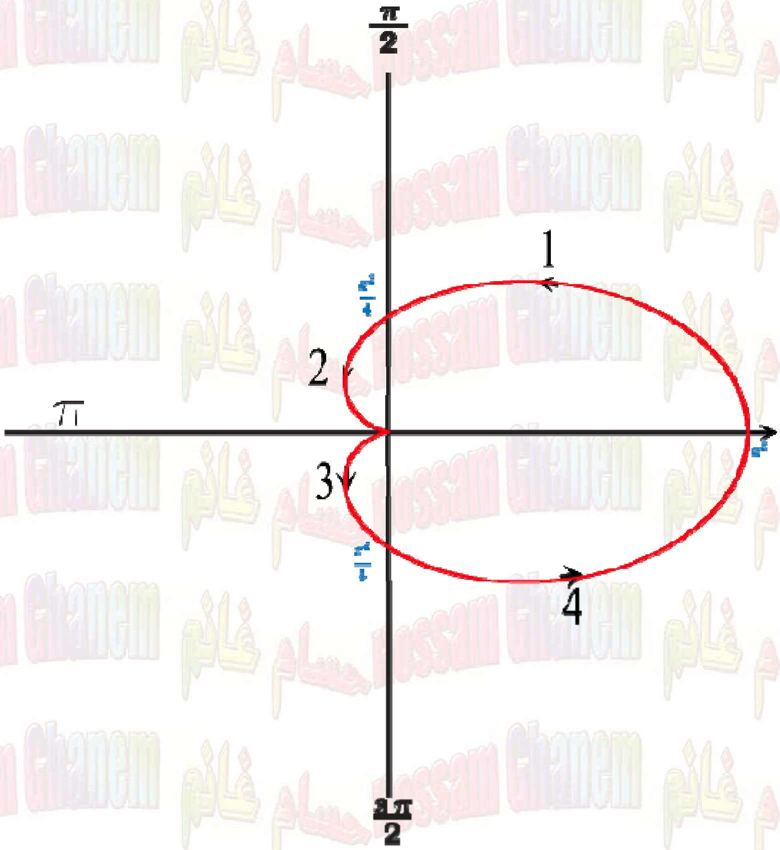
	θ	r
1	$0 \to \frac{\pi}{2}$	$\pi^2 \to \frac{\pi^2}{4}$
2	$\frac{\pi}{2} \to \pi$	$\frac{\pi^2}{4} \to 0$
3	$\pi \to \frac{3\pi}{2}$	$0 \to \frac{\pi^2}{4}$
4	$\frac{3\pi}{2} \to 2\pi$	$\frac{\pi^2}{4} \to \pi^2$

$$A = \frac{1}{2}\int_a^b r^2 d\theta$$

$$A = 2 \cdot \frac{1}{2}\int_0^\pi (\pi - \theta)^4 d\theta$$

$$= \frac{1}{5}(-1)\left[(\pi - \theta)^5\right]_0^\pi$$

$$= \frac{-1}{5}(0 - \pi^5) = \frac{\pi^5}{5}$$



Homework

1	(4 pts.) Find the length of the curve given by $r = \cos^2(\theta/2)$ for $\theta \in [\,0\,,2\pi]$	36 June 6, 2010